Long-range self-affine correlations in a random soliton gas

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The random sine-Gordon chain experiences a sharp crossover to an ordered state associated with the activation of a soliton gas. The spatial coherence of the stationary regimes has three well-defined scaling behaviors. At larger scales there is no correlation, reflecting the independence between the different local structures in the chain. In this paper we introduce a different potential that produces solitons that exhibit long-range interactions. We show that a gas of such solitons can extend self-affinity to all scales. [S1063-651X(97)03206-6]

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For a variety of systems, the interplay between nonlinearity and strong external noise results in fascinating noise-induced transitions to a more structured behavior [1–3]. In particular, roughening transitions in random media have been the object of many studies, due to the interdisciplinary aspects of the problem [4].

In this paper we study noise-induced roughening transitions for random soliton gases generated by Klein-Gordon equations:

$$\nabla^2\phi-\phi_{tt}+G(\phi)=\alpha\phi_t-R(x,t). \tag{1}$$

Here, $G(\phi)=-\partial U(\phi)/\partial\phi$, $U(\phi)$ is a nonlinear function that possesses two or more minima and $R(x,t)$ is spatiotemporal white noise that can represent thermal driving. Many systems [2,5–28] are described by the Klein-Gordon equations including charge density waves [11,12,24], Josephson junctions [2,20–22], structural phase transitions [5–8], crystal growth [17,28], polymers [9,10,19], escaping processes [2], chain dynamics [29], chemical reactions [2], proton conductivity, macromolecules, and hydrogen-bond chains [13,14]. One of the more-studied, particular cases of Eq. (1) is the random sine-Gordon equation [in this case, $G(\phi)=-\sin(\phi)$], which models polynuclear crystal growth if the solution $\phi(x,t)$ is considered the height of a one-dimensional surface [17]. This model exhibits noise-induced pattern formation [30] and the random soliton gas has been related with the roughening (ζ) and dynamic exponents ($\beta=\zeta/z$) [31–33]. Also, its dynamics has been related with the Kardar-Parisi-Zhang (KPZ) [34] and Sneppen [35] universality classes. Before the onset of the noise-induced transition to the soliton bearing regime, the roughening exponent is zero. After the activation of solitons, there is a very interesting crossover from non-KPZ behavior ($\zeta\sim0.7-0.8$) to KPZ behavior ($\zeta\sim0.5$); additionally, for sufficiently large scales, a crossover to a zero-roughening exponent takes place. The $\zeta\sim0$ plateau is more sharply defined when the size of the system is increased, whereas the self-affine regions preserve their extensions [32]. We have verified that the plateau does not depend upon boundary conditions. For the transient, the common dynamic exponent ($\beta\sim0.9$) has been calculated from flat initial conditions for all these regimes. This last result reveals that the surface grows faster than is predicted by the KPZ model and has been related with the global dynamics characteristic of the Sneppen universality class [33]. The dynamics of the random sine-Gordon model is not capable of eliminating disorder at larger scales (for which zero-roughening exponents take place) because this is precisely generated by the different independent and coherent behaviors at small scales. In the random sine-Gordon model, the interactions between the solitons decay exponentially [$F\sim\exp(-d)$] and therefore larger scales exhibit no correlation. The overdamped regime (in presence of a constant driving force) of the random sine-Gordon model has been the object of theoretical and numerical studies [17,36,37].

In this paper we present an alternative model, for which self-affinity extends to all scales. This alternative model takes advantage of the fact that, for a potential with non-Morse critical points and whose first term different from zero in its Taylor expansion around the minima is of order $2n$ (i.e., $U(\phi)\sim\phi^{2n}$, $n>1$), the interaction force decreases with distance as $F\sim d^{2n/(1-n)}$ [18]. In this case, solitons possess long-range interactions and the dynamics of Eq. (1) presents outstanding differences with respect to classic sine-Gordon: the KPZ scaling extends to higher scales, whereas the non-KPZ slope preserves its range.

Previous works [15,18] have shown that Eq. (1) has solutions of the kink and anti-kink types when the potential $U(\phi)$ possesses two or more minima. For $n=1$, the asymptotic behavior of the solitons is exponential and the interaction force F between two solitons decays exponentially with the distance d [18]. In contrast, when $n>1$, the behavior for $x\to\infty$ is $\phi-\phi_j\sim x^k$ (here, $k=1/1-n$ and ϕ_j

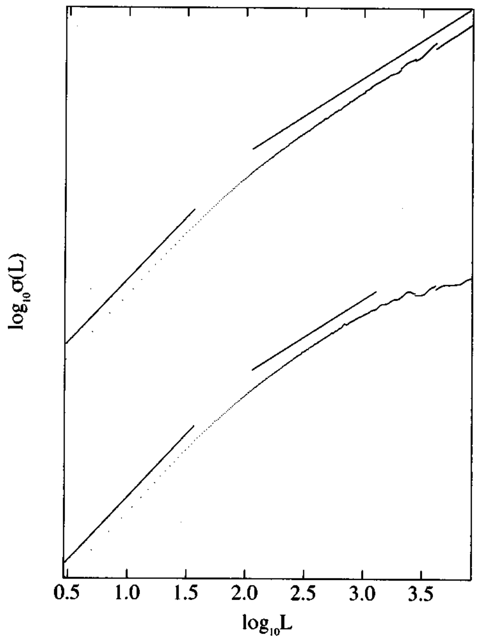

FIG. 1. Stationary regimes for short- and long-range models ($\alpha = 0.252$): log(σ) vs log(L) for $n = 40$ (upper curve) and $n = 1$ (lower curve). There is no scale at the vertical axis because we have shifted curves in order to accomodate them in one figure.

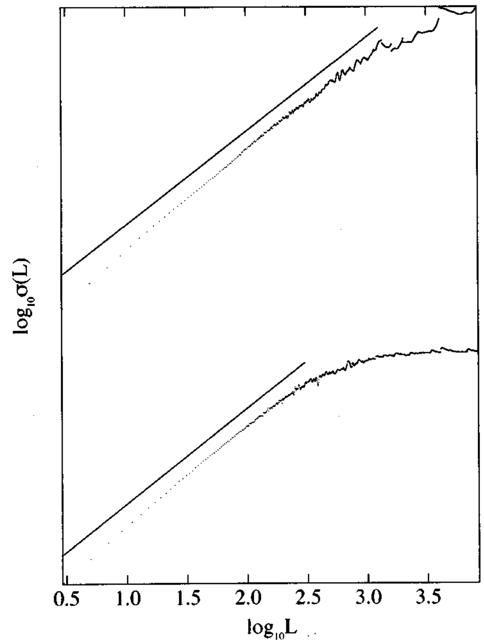

FIG. 2. Overdamped regimes for short- and long-range models ($\alpha = 25.2$): log(σ) vs log(L) for $n = 40$ (upper curve) and $n = 1$ (lower curve).

are the values of two contiguous minima; $j = 1,2$) and the interaction force decays as $F \sim d^{2n/(1-n)}$. Notice that this is a power-law behavior.

Consider the model given by the following dimensionless equations:

$$\phi_{xx} - \phi_{tt} - n \sin\phi (1 - \cos\phi)^{n-1} = \alpha \phi_t - R(x,t), \tag{2}$$

$$\langle R(x,t) \rangle = 0, \tag{3}$$

$$\langle R(x,t) R(x',t') \rangle = 2D\delta(t-t')\delta(x-x'). \tag{4}$$

Here, $U(\phi) = (1 - \cos\phi)^n \equiv 2\sin^{2n}(\phi/2)$. This potential can describe an anharmonic oscillator with a particle moving in a potential of thin and periodic walls, such that the particle almost does not interact with the walls unless it is very close to them [38]. The stability of the particle in the created wells has a marginal character because these are degenerate fixed points of the corresponding dynamical system. Considering a chain of these oscillators can lead to an equation like Eq. (2), which can be used as a growth model in periodic media with marginal stability. For $n = 1$ Eq. (2) reduces to the sine-Gordon equation and for $n > 1$ its soliton solutions exhibit power-law interactions. This model can be associated with recently observed algebraic solitons with long-range interaction and power-law behaviors [27,39–43]. We use flat initial conditions [$\phi(x,0) = \phi_t(x,0) = 0$] and open-boundary conditions [$\phi_x(0,t) = \phi_x(l,t) = 0$]. The parameters of our simulations are $\Delta x = 0.039$, $\Delta t = 0.035$, $l = 320$, and the variance of the noise is 3,333.3 (which is well above the onset of the random soliton gas); we discretize the equation into 8192 points.

In Fig. 1, we present the scaling behavior of the length of the ensemble average of the standard deviation of the height of the spatiotemporal profile [$\sigma(L,t) \sim L^{\zeta}$] for the low-dissipation regime ($\alpha = 0.252$). The lower curve of Fig. 1 corresponds to the stationary regime for the random sine-Gordon case ($n = 1$, $\alpha = 0.252$); this curve reveals two different scaling behaviors $\sigma(L) \sim L^{\zeta}$, in particular, a KPZ behavior ($\zeta \sim 0.50$) for intermediate scales. For small length scales, $\zeta = 0.826 \pm 0.002$, whereas $\zeta = 0.493 \pm 0.001$ for intermediate scales. For larger scales a crossover to a zero-

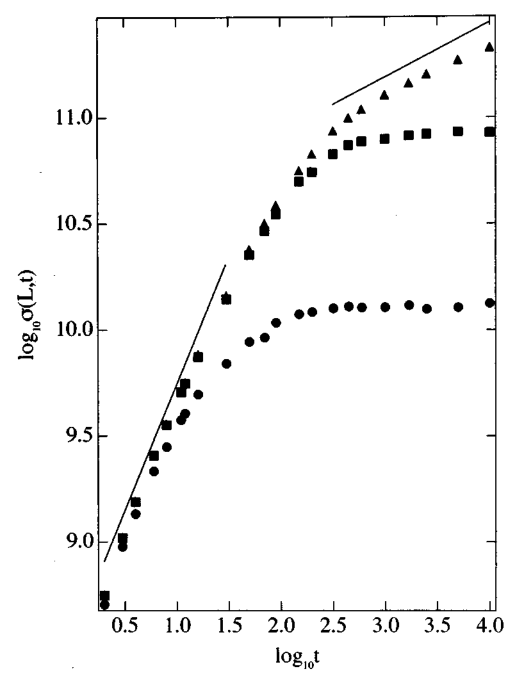

FIG. 3. Transient regime for the long-range model: log(σ) vs log(t) for $\alpha = 0.252$ and $n = 40$.

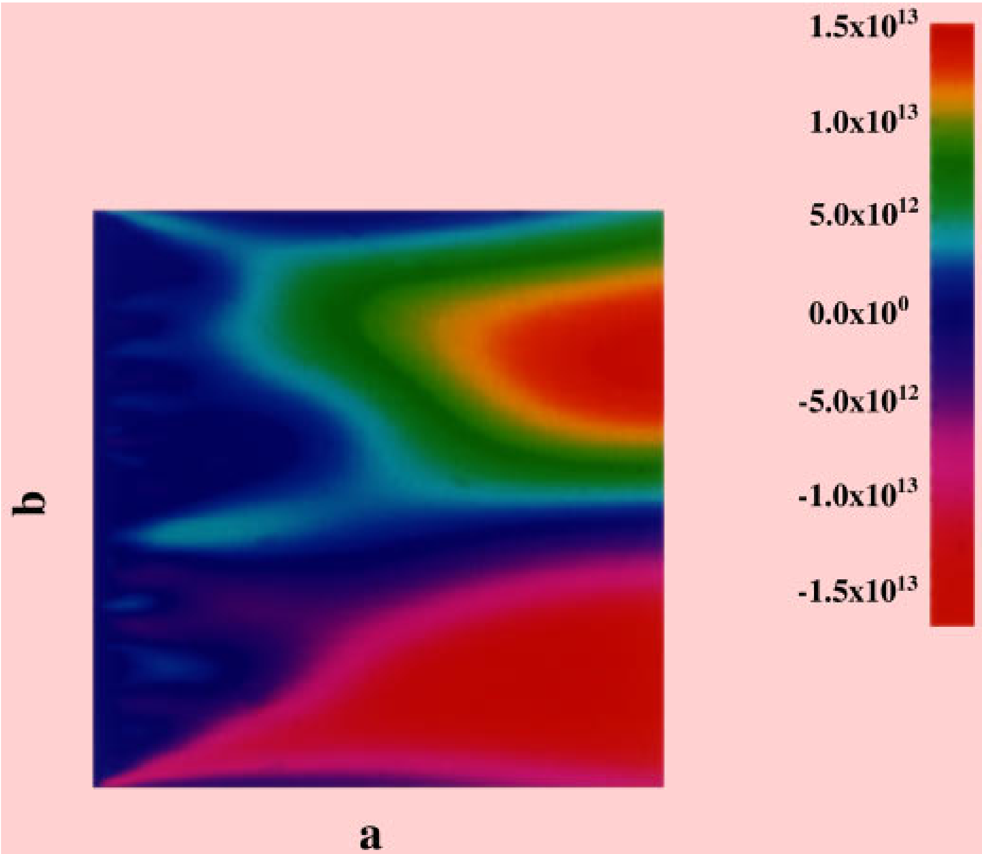

FIG. 4. (Color) Wavelet decomposition for the long-range model ($\alpha=0.252$ and $n=40$).

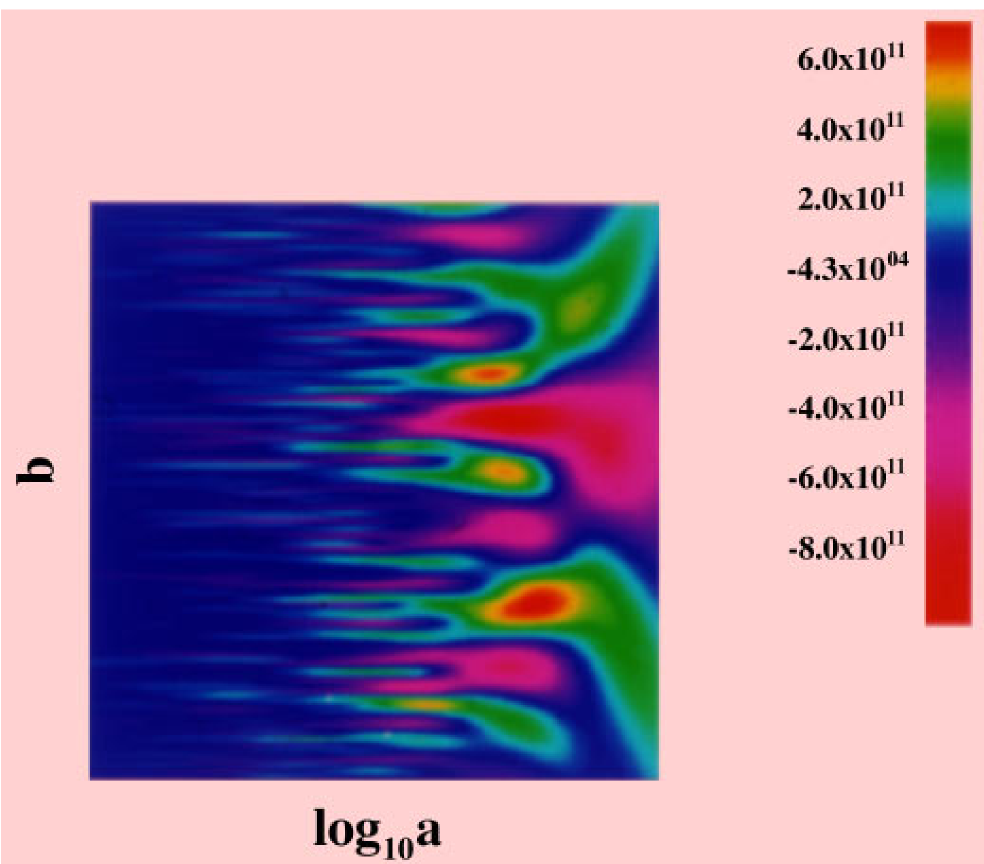

FIG. 5. (Color) Wavelet decomposition for the long-range model ($\alpha=25.2$ and $n=40$). Logarithmic scale (base 10) reveals pitchfork bifurcations.

roughening exponent takes place. The upper curve in Fig. 1 corresponds to $n=40$ ($\alpha=0.252$). In contrast with the random sine-Gordon case, the surface for $n=40$ exhibits only two self-affine regimes, the anomalous ($\zeta=0.818\pm0.002$) and the KPZ-like ($\zeta=0.4977\pm0.0004$). There is statistical fractality (we average over 5000 realizations for each size L) at all the scales of the system, even for the larger ones that the random sine-Gordon equation is not able to order.

Figure 2 is analogous to 1, but for a higher dissipation case ($\alpha=2.52$). We obtain $\zeta=0.490\pm0.002$ for the $n=1$ case and $\zeta=0.481\pm0.001$ for the $n=40$ case. Note that for $n=40$ there is no plateau.

Dynamic scaling theory [4] predicts that the ensemble average of the standard deviation of the spatiotemporal profile satisfies the relation $\sigma(t,L)=L^{\zeta}f(t/L^{z})$. For $t\gg L^{z}$, the system achieves the stationary regime and $\sigma(t,L)=\sigma(L)\sim L^{\zeta}$ [i.e., $f(x)\sim\mathrm{const}$]. For the transient ($t\ll L^{z}$), the system verifies the relation $\sigma(t,L)=\sigma(t)\sim t^{\zeta/z}=t^{\beta}$ [i.e., $f(x)\sim x^{\zeta/z}$]. A high dynamic exponent ($\beta=\zeta/z$) indicates that the progress towards the temporally stable regime is realized very fast, whereas a very low dynamic exponent reveals a slow saturation for the system. In Fig. 3, we plot $\log[\sigma(L,t)]$ versus $\log(t)$ for $n=40$ and $\alpha=0.252$ (we average over 500 realizations from flat initial conditions). The three curves correspond to small, intermediate, and large lengths in the chain. We find that, for sufficiently small values of t, the three curves give a scaling $\sigma(L,t)\sim t^{\beta}$ with $\beta=1.19\pm0.03$. Notice that $\beta>1$ indicates a violent dynamics (almost ballistic). As the system evolves, the curves begin to separate and for the shortest lengths the stationary regime is reached early. For L corresponding to larger scales, the dynamic exponent exhibits a crossover to $\beta=0.26\pm0.01$. For $n=1$ we find, for small values of t, a scaling with $\beta=0.952\pm0.008$.

We resort to the wavelet tranform analysis [44] in order to unveil the structures present in the $\phi(x,t)$ profiles. Figure 4 presents the wavelet decomposition of a stationary state profile for $n=40$. Here a and b are the scale and position parameters, respectively; we had chosen the so-called Mexican hat as analyzing wavelet [44]. This figure reveals the presence of coherent structures at different scales. If we repeat the wavelet decomposition for $n=1$ with the same color scales, we would observe a completely blue picture, showing a relatively moderate growth of the surface for lower resolution (higher scales), whereas $n=40$ exhibits huge values. The existence of an increased coherence in the geometry of the surface for higher scales for the $n=40$ case is evident.

The wavelet transform also reveals the local self-similarity of fractal objects. Figure 5 shows the wavelet decomposition for the overdamped regime of the long-range soliton gas. Logarithmic scale (base 10) allows for appreciation of pitchfork bifurcations associated with fractal order.

We must stress that in the neighborhood of the point $\phi=0$ (as well as in the neighborhood of the other minima), the potential $U(\phi)$ (for $n\gg1$) behaves as a flat well in which $U(\phi)\approx0$ for all the points in the neighborhood. Equation (2) for $n>1$ has no mass term. This fact leads to power-law behaviors [18,27,45]. Something similar occurs in other models for which self-organized criticality has been reported. Moreover, Eq. (2) possesses a nonlinearity in ϕ, such that $G(\phi)$ changes sign, allowing the conditions for the existence of kink and anti-kink solutions. All these features makes the long-range Klein-Gordon model special: it bears solitons that interact with long-range forces, and the system, when forced at random, presents fractal behavior at all scales. Up to our knowledge, there is no system with all these features present at once.

Note that the numerical experiments were inspired by an exact theoretical result concerning the interaction force between the solitons, which (when applied to our model) allowed the prediction of the existence of power-law behaviors in the system. The same theoretical considerations permitted us to foresee strong differences between the cases $n=1$ and $n\gg1$.

The numerical results are in agreement with these considerations. Besides, we used two different (and independent) investigation tools (namely, roughening exponents and wavelet transform) that showed noncontradictory results.

Equation (1), with dimension $D>1$ and the long-range interaction properties that we have already presented, is a very promising *model system*, with applications in a variety of physical systems. This model system should exhibit structures at all scales due to the formation of spontaneous topological defects with long-range interactions, which can create bound states, clusters of bound states, clusters of clusters, etc. In the same manner, a potential in which topological defects possess such behavior can be constructed for other equations, e.g., the complex Ginzburg-Landau equation [46]. Models with these properties have been intensively searched [47–49] due to its importance in the description of complex dynamics, in the presence of vortices, spiral waves, cosmic strings, etc.

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